Segre classes and Damon–Kempf–Laksov formula in algebraic cobordism

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Abstract

In this paper we extend to algebraic cobordism the classical Damon–Kempf–Laksov formula, which expresses the Chow ring Schubert classes of Grassmann bundles as Schur determinants in Chern classes. The basic building block of our closed formula, which is written as a sum of determinants, is represented by a generalisation of Segre classes that we introduce and describe.

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1. Introduction

The classical Damon–Kempf–Laksov formula describes the Schubert classes associated to Grassmann bundles as Schur determinants in Chern classes. As a direct consequence of this universal setting, one is able to deduce an expression describing the fundamental classes of degeneracy loci of morphisms of vector bundles, provided their codimension is the expected one. The goal of this paper is to generalise this closed determinantal formula from the Chow ring to algebraic cobordism, the universal oriented cohomology theory. To this aim, we will also introduce and describe generalised Segre classes for virtual bundles, which will constitute the entries of the determinants appearing in the main theorem.

Let us now briefly recall the original setting. Let X be a smooth quasi-projective variety, E a bundle of rank e endowed with a full flag $F^\bullet = (0 = F^e \subset \cdots \subset F^1 \subset F^0 = E)$ and set $F_\ell := E/F^\ell$. Then, inside the Grassmann bundle $\mathrm{Gr}_d(E)$ of d-dimensional planes one has Schubert varieties X_λ, indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of length at most d and such

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that $\lambda_1 \leqslant e-d$. Such varieties are obtained by selecting the points of $\mathrm{Gr}_d(E)$ over which the tautological d-dimensional bundle U_d meets the reference flag $F^\bullet$ according to a certain intersection pattern defined by λ. In the Chow ring $CH^*(\mathrm{Gr}_d(E))$, their fundamental classes are given by the Schur determinants

$$[X_\lambda]_{CH} = \Delta_\lambda\Big(c(1),\dots,c(r)\Big) := \det\Big(c(i)_{\lambda_i+j-i}\Big)_{1\leqslant i,j\leqslant r}, \tag{1.1}$$

where $c(i)$ denotes the total Chern class $c(F_{\lambda_i+d-i}-U) = c(F_{\lambda_i+d-i})/c(U)$.

Ever since the introduction by Levine and Morel in [4] of algebraic cobordism, denoted Ω^*, a lot of effort has been devoted to lifting results of Schubert calculus to this more general setting. In fact, algebraic cobordism is universal among oriented cohomology theories, a family of functors which includes both the Chow ring and $K^0[\beta,\beta^{-1}]$, a graded version of the Grothendieck ring of vector bundles: this implies that formulas that hold in Ω^* specialise to all other theories. A careful inspection of the original proofs, in which $[X_\lambda]_{CH}$ is computed by pushing forward to $\mathrm{Gr}_d(E)$ the fundamental class of a resolution of singularities of X_λ denoted

$$\psi : Y_\lambda \to X_\lambda \hookrightarrow \mathrm{Gr}_d(E),$$

convinced us that it is more natural to express (1.1) in terms of the Segre classes s(-). Given that for any bundle V one has $c(V) = s(-V^\vee)$, this alternative formulation reads

$$[X_\lambda]_{CH} = \psi_*[Y_\lambda]_{CH} = \Delta_\lambda\Big(s(1),\dots,s(r)\Big) \tag{1.2}$$

where $s(i)$ represents the total Segre class $s\big((U-F_{\lambda_i+d-i})^\vee\big)$.

In this format the formula does generalise, provided that one introduces an appropriate notion of Segre classes, denoted $\mathscr{S}$, and a power series $P(z,x)$, defined as the unique solution to the equation $F(z,\chi(x)) = (z-x)P(z,x)$. Here F represents the universal formal group law on the Lazard ring $\mathbb{L}$ and χ its formal inverse. More precisely, we obtain the following result.

THEOREM A (*cf.* Theorem 5.7). *For any partition $\lambda \in \mathcal{P}_d(n)$ of length r, we have*

$$[Y_\lambda \to \mathrm{Gr}_d(E)]_\Omega := \psi_*[Y_\lambda]_\Omega = \sum_{\mathbf{s}\in\mathbb{N}^r} a_{\mathbf{s}}\Delta_{\lambda+\mathbf{s}}\Big(\mathscr{S}(1),\dots,\mathscr{S}(r)\Big), \tag{1.3}$$

where $\mathscr{S}(i) = \mathscr{S}\big((U-F_{\lambda_i+d-i})^\vee\big)$ and the coefficients $a_{\mathbf{s}} \in \mathbb{L}$ are given by

$$\prod_{1\leqslant i<j\leqslant r} P(t_j,t_i) = \sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{N}^r} a_{\mathbf{s}}\cdot t_1^{s_1}\cdots t_r^{s_r}.$$

When this formula is specialised to the Chow ring one recovers (1.2) and, moreover, the two notions of Segre classes actually coincide. For another example, involving formal group laws given by polynomials, we refer the reader to [5] in which we described more explicitly the case of infinitesimal cohomology theories.

To better appreciate the significance of our formula, it may be useful to place it within the wider framework of generalised Schubert calculus. In the context of algebraic geometry this line of research was pioneered by Calmés–Petrov–Zanoulline ([6]) and Hornbostel–Kiritchenko ([7]) who, inspired by the work of Bressler–Evens on topological cobordism ([8, 9]), studied the algebraic cobordism of flag manifolds. Later, the attention shifted to flag bundles with contributions given by Kiritchenko–Krishna ([10]), Calmes–Zainoulline–Zhong ([11]) and the first author ([12, 13]).

It should be noticed, however, that the interpretation of such results requires a little caution. On the one hand not all Schubert varieties have a well defined notion of fundamental class, only those that are local complete intersection schemes. On the other hand, all classical techniques inherently depend on the choice of a resolution, which in general is not unique.

Following [14], the key idea behind our result is to combine the geometric input given by the Damon–Kempf–Laksov resolution (see Definition 5.1) with an algorithmic procedure modelled after the one used by Kazarian in [15] to describe the Schubert classes of maximal isotropic Grassmann bundles of symplectic and orthogonal type. To be able to use Kazarian's machinery, we had to introduce a suitable notion of Segre classes for Ω^* and find a way to describe them as explicitly as possible. The definition we use is the exact analogue of the one given by Fulton in [16]. We then manage to relate the resulting Segre polynomial $\mathscr{S}(V;u)$ to the Chern polynomial $c(V;u)$ through the formula

$$\mathscr{S}(V;u)c(V;-u) = \frac{\mathscr{P}(u)}{w(V;u)},$$

where $\mathscr{P}(u) := \sum_i [\mathbb{P}^i]_\Omega \cdot u^{-i}$ and $w(V;u)$ is a series constructed by making use of P (see Definition 4.2). An important consequence of this description is that it allows to lift the definition of Segre classes to the Grothendieck ring of vector bundles, so that they can be evaluated on virtual bundles. The following result provides a geometric interpretation to such extension.

Theorem B (*cf.* Theorem 4.9). *Let V and W be two vector bundles over X, respectively of rank n and m. Consider the dual projective bundle $\mathbb{P}^*(V) \xrightarrow{\pi} X$ with tautological bundle $O(1)$. Then one has*

$$\mathscr{S}_{m-n+1}(V-W) = \pi_*\Big(c_m\big(O(1)\otimes W^\vee\big)\Big)$$

as elements of $\Omega^*(X)$.

After our work was completed, we were informed by Nakagawa–Naruse that in [17] they achieved, by considering a different resolution, a stable generalisation of the Hall–Littlewood type formulas for Schur polynomials in the context of topological cobordism (*cf.* [18]). Finally, it is known from the work of Lascoux–Schutzenberger [19] and Fulton [3] that in cohomology one can express the degeneracy loci classes associated to *vexillary* permutations as determinants (*cf.* Anderson–Fulton [20]). In [21] we generalised this result to both connective K-theory and Ω^*.

Notations and conventions: In this paper $\mathbf{k}$ stands for a field of characteristic 0 and $\mathbf{Sm_k}$ is the category of smooth separated schemes of finite type which are quasi-projective over $\mathrm{Spec}(\mathbf{k})$. Finally, we will follow the convention according to which 0 belongs to the natural numbers $\mathbb{N}$.

2. Degeneracy loci and their fundamental classes

The main goal of this section is to relate the classical Damon–Kempf–Laksov formula with the broader theory of degeneracy loci. We will also highlight how the properties of the different families of resolutions of singularities of Schubert varieties are reflected in the formulas. In doing so we will explain which modifications are needed for the generalisation to Ω^*. These will be carried out in detail in sections 4 and 5, after a brief revision of algebraic cobordism in section 3.

2.1 A recursive formula

Let us begin by recalling the definition of $D_r(V,W)$, the degeneracy locus of a morphism of vector bundles $\varphi : V \to W$ over a base scheme X. For $r \leqslant \operatorname{rank} V, \operatorname{rank} W$ the closed subscheme $D_r(V,W) \subseteq X$ is defined as the locus of points over which the dimension of the image of φ is at most r, with the scheme structure being given by the vanishing of induced morphisms on the $(r+1)$-th exterior powers.

The most general result describing the fundamental classes of degeneracy loci is due to Fulton [3, Theorem 8.2] and it considers morphisms between flags of vector bundles. Suppose to be given, together with φ, the following flags of subbundles: $V_\bullet = (0 = V_{p_0} \subset V_{p_1} \subset \cdots \subset V_{p_s} = V)$ and $W_\bullet = (W = W_{q_t} \twoheadrightarrow \cdots \twoheadrightarrow W_{q_1} \twoheadrightarrow W_{q_0} = 0)$. The reader should be aware that in this section, unlike the rest of the paper, subscripts will denote the rank of the associated bundle. If for every pair of indices (i,j) we are given rank numbers $r_{i,j}$, we can consider the degeneracy loci $D_{r_{i,j}}(V_{p_i}, W_{q_j})$. Under appropriate assumptions on the collection $\mathbf{r}$ of the rank numbers, *i.e.* that it is possible to associate to it a permutation $w(\mathbf{r})$, Fulton proved a formula describing the fundamental class of the intersection $D_{\mathbf{r}}(V_\bullet, W_\bullet) := \bigcap_{(i,j)} D_{r_{i,j}}(V_{p_i}, W_{q_j})$. In fact, provided that $D_{\mathbf{r}}(V_\bullet, W_\bullet)$ has the expected codimension, its Chow ring class in $CH^*(X)$ is given by

$$[D_{\mathbf{r}}(V_\bullet, W_\bullet)]_{CH} = \mathfrak{S}_{w(\mathbf{r})}(x_1, \ldots, x_{q_t}, y_1, \ldots, y_{p_s}), \tag{2.1}$$

where $\mathfrak{S}_{w(\mathbf{r})}$ is the double Schubert polynomial associated to $w(\mathbf{r})$ and the x_i's and the y_i's are, respectively, the Chern roots of W and V. Double Schubert polynomials are a family of polynomials with integer coefficients in two sets of variables indexed by permutations. They are defined recursively through the application of the divided difference operators ∂_i, also known as symmetrising operators. As a direct consequence of this recursive definition and of the special properties of $w(\mathbf{r})$ one has that $\mathfrak{S}_{w(\mathbf{r})}$ is unchanged by the action on the variables of the product of symmetric groups $S_{q_1-q_0} \times \cdots \times S_{q_t-q_{t-1}} \times S_{p_1-p_0} \times \cdots \times S_{p_s-p_{s-1}}$. This implies that (2.1) can be rewritten in terms of the Chern classes of the bundles $A_i := V_{p_i}/V_{p_{i-1}}$ and $B_i := \mathrm{Ker}(W_{q_i} \twoheadrightarrow W_{q_{i-1}})$ as

$$[D_{\mathbf{r}}(V_\bullet, W_\bullet)]_{CH} = P_{\mathbf{r},\mathbf{p},\mathbf{q}}\Big(c(B_1), \ldots, c(B_t), c(A_1), \ldots, c(A_s)\Big) \tag{2.2}$$

for some polynomial $P_{\mathbf{r},\mathbf{p},\mathbf{q}}$. At this stage it is important to stress two aspects. The first is that (2.2) is not closed or, in other words, although one knows that the polynomial $P_{\mathbf{r},\mathbf{p},\mathbf{q}}$ has to exist (in view of the fact that the associated double Schubert polynomial presents a certain type of simmetry), still one does not know a priori what its coefficients are going to be. The second observation is that the recursive nature of the answer mirrors the recursive definition of the geometric objects involved in the proof (*i.e.* the Bott–Samelson resolutions).

2.2 A determinantal formula

On the other hand there exists a family of permutations, known as vexillary, for which it is possible to establish closed determinantal formulas describing their associated double Schubert polynomials. This of course implies the existence of closed formulas for the fundamental classes of those degeneracy loci whose permutations happen to belong to this family. Although it is possible to derive these expressions purely algebraically, from a geometric perspective their existence is due to the presence of a different family of resolutions of singularities, which we refer to as Damon–Kempf–Laksov. By making use of these resolutions Damon ([1]) and Kempf–Laksov ([2]) proved that, in the special case in which $s \geqslant p_s - q_t$, $t = 1$ and $r_{i,1} = p_i - i$ for $i \in \{1, \ldots, s\}$, one has the following determinantal expression

$$[D_{\mathbf{r}}(V_\bullet, W_\bullet)]_{CH} = \Delta_\lambda\Big(c(W - V_{p_1}), \ldots, c(W - V_{p_s})\Big) := \det\Big(c_{\lambda_i+j-i}(W - V_{p_s})\Big)_{1\leqslant i,j\leqslant r}, \tag{2.3}$$

where $\lambda = (q_t - p_1 - 1, \ldots, q_t - p_s - s)$. It is worth noticing that, although this formula was later derived algebraically from (2.1) by Fulton, the original proof is completely independent and more geometric in nature.

Let us briefly outline its structure. First of all, by standard methods one reduces the problem to the universal case represented by a Schubert variety of the Grassmann bundle $\mathrm{Gr}_s(V\oplus W)$. For each Schubert variety X_λ one constructs its Damon–Kempf–Laksov resolution Y_λ, which is itself a Schubert variety in a partial flag bundle $\pi : \mathrm{Fl}_s \to \mathrm{Gr}_s(V \oplus W)$. In $CH^*(\mathrm{Fl}_s)$ the fundamental class $[Y_\lambda]$ can be computed easily and one has to perform some Gysin computations to obtain a determinantal expression of $\pi_*[Y_\lambda]_{CH} = [X_\lambda]_{CH}$ in terms of the relative Segre classes $s(W - V_{p_j})$. Finally, one rewrites the resulting formula using Chern classes.

It is this proof that we want to lift to Ω^*, focusing exclusively on the universal case of the Schubert varieties of Grassmann bundles. It should be noticed that Ω^* admits a notion of fundamental class only for local complete intersection schemes and that not all Schubert varieties X_λ belong to this family. As a consequence we set as our goal to compute the pushforward classes $\pi_*[Y_\lambda]_\Omega$, which always exist and are uniquely associated to every X_λ.

To be able to modify to algebraic cobordism the classical Gysin computation, we first need to generalise the notion of relative Segre classes. Once this has been done, by adapting the classical definition of Segre classes and by identifying an appropriate generating function describing them, one is left with the task of producing the analogue of the determinantal formula. To this end we employ an adaptation of the method used by Kazarian in [15], which reduces the manipulation of complicated expressions involving Segre classes to computations with Laurent series.

2.3 A comparison between different resolutions

As briefly pointed out in our comparison of (2.2) and (2.3), different formulas describing the fundamental class of the same Schubert variety X_λ arise from different geometric pictures and more specifically from different resolutions of singularities. In fact, there exists a third way of expressing these fundamental classes, *i.e.* the Hall–Littlewood formula, behind which lies a family of desingularisations $\varphi : Z_\lambda \to X_\lambda$ known as *small*. The name comes from the requirements imposed on the size of the subsets $\{x \in X_\lambda \mid \mathrm{codim}_Z(\varphi^{-1}(x)) > k\}$, whose codimension in X_λ needs to be at least $2k$. In particular, often these conditions are not satisfied by Bott–Samelson and Damon–Kempf–Laksov resolultions, whose fibers tend in general to be bigger.

An important difference between these families of resolutions is their behaviour with respect to stability. For every vector bundle $E \to X$ one obtains an infinite system of Grassmann bundles $\mathrm{Gr}_d(E \oplus O_X^n)$, each embedded in the next, and the Schubert varieties X_λ are compatible with respect to the pullback maps arising from the embeddings. Since the construction of both small and Damon–Kempf–Laksov resolutions is unaffected by the dimension of the ambient space in which X_λ lives, the formulas they give rise to are automatically compatible with pullbacks. The same does not hold for Bott–Samelson resolutions. The left hand side of (2.1) happens to be stable in CH^* thanks to the good properties of divided difference operators, but the corresponding statement for Ω^* is not. Actually, for every n it becomes necessary a careful selection among the different Bott–Samelson resolutions associated to X_λ to be able to obtain stable expressions.

Another way in which these three families differ is given by the degree with which the pushforward class can be computed explicitly. While Bott–Samelson and Damon–Kempf–Laksov resolutions are constructed as subsets of towers of, respectively, $\mathbb{P}^1$-bundles and $\mathbb{P}^n$-bundles (with varying n), small resolutions live inside towers of Grassmann bundles. The number of levels of the towers is given by the codimension of the Schubert variety X_λ in the first case, *i.e.* $|\lambda|$, by the number of parts of λ in the second and by the number of different parts in the third. This implies that, when one is taking the pushforward along the different towers, small resolutions require the least number of steps, but on a single step the other two kinds can be computed much more concretely, through explicit closed formulas.

Although it is theoretically possible to obtain closed formulas by exploiting small resolutions, so far these have given rise only to the Hall–Littlewood formula (*cf.* [17]): as with (2.1) one knows that it can be written using Chern classes, but the actual expression is not known. More specifically, so far it has not been possible to find an adequate description of the analogues of Segre classes for Grassmann bundles, nor a procedure like the one of Kazarian.

3. Preliminaries on algebraic cobordism

An oriented cohomology theory consists of a contravariant functor $A^* : \mathbf{Sm_k} \to \mathcal{R}^*$, together with a family of pushforward maps $\{f_* : A^*(X) \to A^*(Y)\}$, one for each projective morphism $f : X \to Y$. We will not describe in detail the compatibilities and the properties that this data is required to satisfy, the interested reader can find the precise definition in [4, Definition 1.1.2]. Instead, we will illustrate the aspects in which a general oriented cohomology theory differs from the Chow ring, the simplest and perhaps best known example, which the reader should always bear in mind as a first approximation.

Since all oriented cohomology theories satisfy the projective bundle formula, each of them allows a theory of Chern classes which, in most respects, mirrors the one for CH^*: to every vector bundle $V \to X$ it is possible to associate a Chern polynomial $c^A(V;u) \in A^*(X)[u]$. Such assignment respects the Whitney formula, so that it can be extended to the Grothendieck group of vector bundles $K^0(X)$. To a class $[V]-[W]$ one associates

$$c^A(V-W;u) = \frac{c^A(V;u)}{c^A(W;u)} \quad \text{or, equivalently,} \quad c_i^A(V-W) = \sum_{j=0}^{i}(-1)^j c_{i-j}^A(V) h_j^A(W), \tag{3.1}$$

where $h_j^A(W)$ stands for the j-th complete symmetric function in the Chern roots of W.

A close examination of the behaviour of the first Chern classes of line bundles unveils a key aspect in which CH^* proves to be too limited to adequately represent all theories. While it is well known that c_1^{CH} is linear with respect to tensor product, this is no longer true in general: describing $c_1^A(L\otimes M)$ in terms of the classes of the factors requires the use of a formal group law $(A^*(\operatorname{Spec}\mathbf{k}), F_A)$. This is a power series F_A defined over the coefficient ring $A^*(\operatorname{Spec}(\mathbf{k}))$ such that, for any choice of line bundles L and M over some scheme X, one has

$$c_1^A(L\otimes M) = F_A(c_1^A(L), c_1^A(M)).$$

In a similar fashion, the usual equation $c_1^{CH}(L^\vee) = -c_1^{CH}(L)$ becomes $c_1^A(L^\vee) = \chi_A\big(c_1^A(L)\big)$, where $\chi_A \in A^*(\operatorname{Spec}(\mathbf{k}))[[u]]$ is the so-called formal inverse, the unique power series such that

$$F_A\big(u, \chi_A(u)\big) = 0.$$

The main achievement of Levine and Morel concerning oriented cohomology theories is the construction of algebraic cobordism, denoted Ω^*, which they identify as universal in the following sense.

THEOREM 3.1 ([4, Theorems 1.2.6 and 1.2.7]). *Ω^* is universal among oriented cohomology theories on $\mathbf{Sm_k}$. That is, for any other oriented cohomology theory A^* there exists a unique morphism*

$$\vartheta_A : \Omega^* \to A^*$$

of oriented cohomology theories. Furthermore, its associated formal group law $(\Omega^*(\mathrm{Spec}(\mathbf{k})), F_\Omega)$ *is isomorphic to the universal one defined on the Lazard ring* $(\mathbb{L}, F)$.

One of the consequences of the universality is that it allows to translate formulas which hold in Ω^* to every other oriented cohomology theory A^*, by making use of ϑ_A. In particular, if the given formula has a classical version in either CH^* or K^0, then one is supposed to recover it. On the other hand, it is not always the case that properties that hold for the Chow ring or the Grothendieck ring will lift to algebraic cobordism.

For example, one basic instance of this pheonomenon can be observed if one tries to compute the fundamental class of some closed subscheme $Z \overset{i_Z}{\hookrightarrow} X$ in a smooth ambient space. While for the Chow ring it is sufficient to consider any resolution of singularities $\widetilde{Z} \xrightarrow{\varphi_{\widetilde{Z}}} X$ to recover $[Z]_{CH}$ as $\varphi_{\widetilde{Z}*}[\widetilde{Z}]_{CH}$, for K^0 one is able to conclude that $[\mathcal{O}_Z]_{K^0} = \varphi_{\widetilde{Z}*}[\mathcal{O}_{\widetilde{Z}}]_{K^0}$ only if Z has at worst rational singularities. Even this weaker statement proves to be false for algebraic cobordism, since different desingularisations can yield different push-forward classes.

On top of this lies an even bigger problem. As mentioned in the introduction, in Ω^* a scheme $Z \xrightarrow{\pi_Z} \mathrm{Spec}(\mathbf{k})$ has a well defined notion of fundamental class only if it is an l.c.i scheme. In fact, since l.c.i. pullbacks are available, one can make use of Ω_*, the homological counterpart of algebraic cobordism which is defined for all quasi-projective schemes. Namely we can set $[Z]_{\Omega_*} := \pi_Z^*(1)$, where 1 is viewed as the multiplicative unit of the coefficient ring $\mathbb{L}$. Then, as an element of $\Omega^*(X)$, the fundamental class of $[Z]_{\Omega^*}$ is given by $i_{Z*}([Z]_{\Omega_*})$, which as a cobordism cycle can be rewritten as $[Z \overset{i_Z}{\hookrightarrow} X]$. It is worth noting that, since $id_{X*} = id_{\Omega^*(X)}$, for $Z = X$ one recovers the original definition for smooth schemes $1_X := [X \xrightarrow{id_X} X]$.

Let us finish this section by warning the reader that we will follow the common practice of writing $[X]_\Omega$ instead of the more precise notation $[X \xrightarrow{\pi_X} \mathrm{Spec}(\mathbf{k})]$ when dealing with the elements of the coefficient ring $\Omega^*(\mathrm{Spec}(\mathbf{k}))$. More generally, the subscript Ω will from now on be omitted and, unless stated otherwise, all classes are to be thought of as cobordism classes. Finally, we will consider the Lazard ring $\mathbb{L}$ as a graded ring in view of the isomorphism with $\Omega^*(\mathrm{Spec}(\mathbf{k}))$. *For the rest of the paper, we will work with algebraic cobordism* Ω^* *and* $F(u,v) \in \mathbb{L}[[u,v]]$ *will stand for the universal formal group law.*

4. Segre classes and relative Segre classes

In this section we first introduce Segre classes for algebraic cobordism and compute their generating function (Theorem 4.6). Then we use such description to define relative Segre classes, which we later describe in Theorem 4.9 as pushforwards of Chern classes along a projective bundle. This will be the main ingredient for the computation of the Damon–Kempf–Laksov classes in Section 5.

4.1 Definition of $w(E;u)$

In order to describe the generating function of Segre classes, we introduce $w_{-s}(E)$, whose definition is based on the following elementary observation.

Lemma 4.1. *There exists a unique power series* $P(z,x) \in \mathbb{L}[[z,x]]$ *of degree* 0 *and constant term* 1 *satisfying*

$$F(z, \chi(x)) = (z - x)P(z,x).$$

Proof. Let us write $F(z, \chi(x)) = \sum_{j=0}^{\infty} Q_j(z, x)$ where each $Q_j(z, x)$ is a homogeneous polynomial of total degree j in z and x. Each $Q_j(z, x)$ becomes 0 if one sets $z = x$, thus it is divisible by $(z - x)$. Therefore the claim holds. $\square$

Definition 4.2. Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be a set of formal variables. For each integer $s \geqslant 0$, we define $w_{-s}(\mathbf{x}) \in \mathbb{L}[[\mathbf{x}]]$ by

$$\prod_{q=1}^{n} P(z, x_q) = \sum_{s=0}^{\infty} w_{-s}(\mathbf{x}) z^s$$

and let $w(\mathbf{x}; u) := \sum_{s\geqslant 0}^{\infty} w_{-s}(\mathbf{x}) u^{-s}$. If $x_1, \dots, x_e$ are interpreted as the Chern roots of a vector bundle V, then we can define $w(V; u) := w(\mathbf{x}; u)$ and $w_{-s}(V) := w_{-s}(\mathbf{x})$.

Since $w_0(\mathbf{x})$ has constant term 1, it is invertible in $\mathbb{L}[[\mathbf{x}]]$. Moreover, an easy computation yields

$$c_n(L \otimes V^\vee) = \prod_{q=1}^{n} F(z, \chi(x_q)) = \sum_{p=0}^{n} (-1)^p c_p(V) z^{n-p} w(V; z^{-1}). \tag{4.1}$$

4.2 Segre classes

Definition 4.3. Let V be a vector bundle of rank n over X. For each $k \in \mathbb{Z}$, consider the dual projective bundle $\pi_m : \mathbb{P}^*(V \oplus O_X^{\oplus m}) \to X$ for some $m \geqslant \max\{0, -k - n + 1\}$ where O_X is the trivial line bundle over X. We then define the degree k *Segre class* $\mathscr{S}_k(V)$ of V by

$$\mathscr{S}_k^{(m)}(V) = \pi_{m*}(\tau^{k+n+m-1}),$$

where τ is the first Chern class of the tautological quotient line bundle $\mathcal{Q}$ of $\mathbb{P}^*(V \oplus O_X^{\oplus m})$.

Remark 4.4. It is easy to see that the definition of $\mathscr{S}_k(V)$ is actually independent of m. In fact, for $m' > m$ one has a canonical inclusion $\iota_m^{m'} : \mathbb{P}^*(V \oplus O_X^{\oplus m}) \hookrightarrow \mathbb{P}^*(V \oplus O_X^{\oplus m'})$, whose associated pushforward map $(\iota_m^{m'})_*$ is just multiplication by $\tau^{m'-m}$. Then, since $\pi_m = \pi_{m'} \circ \iota_m^{m'}$, one has

$$\mathscr{S}_k^{(m)}(V) = \pi_{m*}(\tau^{k+n+m-1}) = \pi_{m'*}\big((\iota_m^{m'})_*(\tau^{k+n+m-1})\big) = \pi_{m'*}(\tau^{k+n+m'-1}) = \mathscr{S}_k^{(m')(V)}$$

and it is therefore possible to remove the superscript (m) from the notation.

Remark 4.5. If V is a line bundle and $m = 0$, we have $\mathcal{Q} = V$ and $\pi = \mathrm{id}_X$, *i.e.* $\mathscr{S}_k(V) = c_1(V)^k$ for all $k \geqslant 0$.

Theorem 4.6. *Let V be a vector bundle of rank n over $X \in \mathbf{Sm_k}$ and $\mathscr{S}(V; u) = \sum_{k\in\mathbb{Z}} \mathscr{S}_k(V) u^k$. Then we have*

$$\mathscr{S}(V; u) = \frac{\mathscr{P}(u)}{c(V; -u) w(V; u)},$$

where we set

$$\mathscr{P}(u) := \sum_{i=0}^{\infty} [\mathbb{P}^i] u^{-i}$$

with $[\mathbb{P}^i] \in \mathbb{L}^{-i}$ being the class of the projective space $\mathbb{P}^i$.

Proof. We will prove

$$\mathscr{P}(u) = c(V; -u) w(V; u) \mathscr{S}(V; u).$$

Let us begin by proving the equalities in degree $-m$, for $m \in \mathbb{N}$. For this we consider the vector bundle $V \oplus O_X^{m+1} \to X$ and its projectivization $\pi_{m+1} : \mathbb{P}^*(V \oplus O_X^{m+1}) \to X$. Let $\mathcal{Q} \to \mathbb{P}^*(V \oplus O_X^{m+1})$ denote its universal quotient line bundle and τ its first Chern class.

The composition of the bundle maps $\pi_{m+1}^* V \hookrightarrow \pi_{m+1}^*(V \oplus O_X^{m+1}) \twoheadrightarrow \mathcal{Q}$ yields a section $s_{m+1} : \mathbb{P}^*(V \oplus O_X^{m+1}) \to \pi_{m+1}^* V^\vee \otimes \mathcal{Q}$. Since $m+1 \geqslant 0$, we can identify the zero locus of this section: $Z(s_{m+1}) \simeq \mathbb{P}^*(O_X^{m+1}) \simeq (\mathbb{P}^m)^* \times_{\mathrm{Spec}(\mathbf{k})} X$. Moreover, as its codimension in $\mathbb{P}^*(V \oplus O_X^{m+1})$ is n, its fundamental class is given by the top Chern class of $\pi_{m+1}^* V^\vee \otimes \mathcal{Q}$. Hence, together with (4.1), we obtain

$$[Z(s_{m+1}) \to \mathbb{P}^*(V \oplus O_X^{m+1})] = c_n(\pi_{m+1}^* V^\vee \otimes \mathcal{Q}) = \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} c_{n-i}(V) w_{i-k}(V) \tau^k.$$

Now we push-forward this equality to $\Omega^*(X)$ and get

$$\begin{aligned}[\mathbb{P}^m] \cdot 1_X &= \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} c_{n-i}(V) w_{i-k}(V) \pi_{m*}(\tau^k) \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} c_{n-i}(V) w_{i-k}(V) \mathscr{S}_{k-n-m}(V),\end{aligned}$$

which is precisely the desired equality.

Let us now focus on the equalities in degree m, with m strictly positive. To do this we consider the projective bundle $\pi : \mathbb{P}^*(V) \to X$ and the following short exact sequence of vector bundles:

$$0 \to \mathcal{Q}^\vee \to \pi^* V^\vee \to H^\vee \to 0$$

over $\mathbb{P}^*(V)$. By twisting it by $\mathcal{Q}$, we see that the first term is trivial and as a consequence we get $c_n(\pi^* V^\vee \otimes \mathcal{Q}) = 0$. As in the previous part, we expand the left hand side by means of the Chern polynomials and of the power series $w(V,u)$. Hence we obtain

$$c_n(\pi^* V^\vee \otimes \mathcal{Q}) = \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} c_{n-i}(V) w_{i-k}(V) \tau^k,$$

where we set $\tau := c_1(\mathcal{Q})$. It now suffices to multiply both sides by τ^m and push them forward to $\Omega^*(X)$ to obtain the desired equality in degree m. □

Example 4.7. For connective K-theory CK^*, we have $\mathscr{P}(u) = \frac{1}{1-\beta u^{-1}}$ and $w(V;u) = \frac{1}{c(V;-\beta)}$. Thus Theorem 4.6 gives

$$\mathscr{S}(V;u) = \frac{1}{1-\beta u^{-1}} \frac{c(V;-\beta)}{c(V;-u)},$$

which was obtained in [14]. Note that the sign convention for β is opposite from the one in [14]. If, furthermore, we set $\beta = 0$, one then obtains $\mathscr{P}(u) = w(V;u) = 1$ and the corresponding statement for CH^* becomes

$$\mathscr{S}(V;u) c(V;-u) = 1,$$

which is precisely the classical identity that allows one to identify Segre classes with the complete elementary symmetric functions in the Chern roots.

4.3 Relative Segre classes

Let $0 \to V' \to V \to V'' \to 0$ be a short exact sequence of vector bundles. From Definition 4.2, we can observe that

$$w(V;u) = w(V';u)w(V'';u).$$

This allows us to define the following.

Definition 4.8. Let V and W be arbitrary vector bundles over X. For each $s \geqslant 0$, we define the class $w_{-s}(V-W)$ in $\Omega^{-s}(X)$ by

$$w(V-W;u) := \sum_{s=0}^{\infty} w_{-s}(V-W)u^{-s} := \frac{w(V;u)}{w(W;u)}.$$

For each $k \in \mathbb{Z}$, we define the relative Segre class $\mathscr{S}_k(V-W)$ in $\Omega^k(X)$ by

$$\mathscr{S}(V-W;u) := \sum_{k\in\mathbb{Z}} \mathscr{S}_k(V-W)u^k := \frac{\mathscr{P}(u)}{c(V-W;-u)w(V-W;u)}, \tag{4.2}$$

or equivalently,

$$\mathscr{S}_k(V-W) := \sum_{q=0}^{\mathrm{rk}(W)} \sum_{j=0}^{\infty} (-1)^q c_q(W) w_{-j}(W) \mathscr{S}_{k-q+j}(V).$$

Both the classes $w_{-s}(V-W)$ and $\mathscr{S}_k(V-W)$ are well-defined if $[V-W]$ is viewed as an element of the Grothendieck group of vector bundles over X.

The following description generalises Proposition 2.12 of [14] to algebraic cobordism.

Theorem 4.9. *Let V and W be vector bundles over X of rank n and m respectively. Let $\pi : \mathbb{P}^*(V) \to X$ be the dual projective bundle, $\mathcal{Q}$ its tautological quotient line bundle, and $\tau := c_1(\mathcal{Q})$. For any nonnegative integer s, we have*

$$\pi_*(\tau^s c_m(\mathcal{Q}\otimes W^\vee)) = \mathscr{S}_{m-n+1+s}(V-W).$$

Proof. In view of (4.1) one gets

$$\tau^s c_m(\mathcal{Q}\otimes W^\vee) = \sum_{q=0}^{m}\sum_{j=0}^{\infty} (-1)^q c_q(W) w_{-j}(W) \tau^{j+m-q+s}.$$

Thus, by the definition of $\mathscr{S}_k(V)$, we have

$$\pi_*(\tau^s c_m(\mathcal{Q}\otimes W^\vee)) = \sum_{q=0}^{m}\sum_{j=0}^{\infty} (-1)^q c_q(W) w_{-j}(W) \mathscr{S}_{m-n+1+s-q+j}(E),$$

the right hand side of which is $\mathscr{S}_{m-n+1+s}(W-V)$ by (4.2). □

Remark 4.10. If V is a line bundle, then $\pi = \mathrm{id}_X$ as mentioned above. In this case, we have $\tau^s c_m(\mathcal{Q}\otimes W^\vee) = \mathscr{S}_{m-n+1+s}(V-W)$.

5. A determinantal formula for Damon–Kempf–Laksov classes

Let E be a vector bundle of rank e over a smooth quasi-projective variety X. Let $\mathrm{Gr}_d(E) \to X$ be the Grassmann bundle over X consisting of pairs (x, U_x) where $x \in X$ and U_x is a d-dimensional subspace of E_x, the fibre of E at x. Let U be the tautological bundle of $\mathrm{Gr}_d(E)$. Fix a complete flag $0 = F^e \subset \cdots \subset F^1 \subset F^0 = E$ where rk $F^k = e-k$. We set $F_k := E/F^k$. In the rest of the paper we will suppress from the notation the pullback of bundles.

A partition λ with at most d parts is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_d)$ of nonnegative integers. The length of $\lambda \in \mathcal{P}_d$ is the number of nonzero parts, where $\mathcal{P}_d$ is the set of all partitions with at most d parts. Let $\mathcal{P}_d(e)$ be the set of all partitions λ in $\mathcal{P}_d$ such that $\lambda_1 \leqslant e-d$. For each $\lambda \in \mathcal{P}_d(e)$ of length r, we define the degeneracy locus X_λ in $\mathrm{Gr}_d(E)$ by

$$X_\lambda := \left\{(x, U_x) \in \mathrm{Gr}_d(E) \mid \dim(F_x^{\lambda_i+d-i} \cap U_x) \geqslant i, i = 1, \ldots, r\right\}.$$

Consider the r-step flag bundle $\mathrm{Fl}_r(U)$ of U over $\mathrm{Gr}_d(U)$, whose fiber at (x, U_x) is a flag of subspaces $(D_1)_x \subset \cdots \subset (D_r)_x \subset U_x$ with $\dim(D_i)_x = i$. Let $D_1 \subset \cdots \subset D_r$ be the tautological bundles of $\mathrm{Fl}_r(U)$ and set $D_0 = 0$. The flag bundle $\mathrm{Fl}_r(U)$ can be realised as the following tower of projective bundles

$$\begin{aligned}\pi : \mathrm{Fl}_r(U) = \mathbb{P}(U/D_{r-1}) \xrightarrow{\pi_r} \mathbb{P}(U/D_{r-2}) \xrightarrow{\pi_{r-1}} \cdots \\ \cdots \xrightarrow{\pi_3} \mathbb{P}(U/D_1) \xrightarrow{\pi_2} \mathbb{P}(U) \xrightarrow{\pi_1} \mathrm{Gr}_d(E).\end{aligned} \tag{5.1}$$

We regard D_i/D_{i-1} as the tautological line bundle of $\mathbb{P}(U/D_{i-1})$. Denote $\tau_i := c_1((D_i/D_{i-1})^\vee)$.

Definition 5.1. For each $\lambda \in \mathcal{P}_d(n)$ of length r, define a subvariety $Y_\lambda \subset \mathrm{Fl}_r(U)$ by

$$Y_\lambda := \left\{\left(x, U_x, (D_\bullet)_x\right) \in \mathrm{Fl}_r(U) \mid (D_i)_x \subset F_x^{\lambda_i+d-i}, i = 1, \ldots, r\right\}.$$

The cobordism class $[Y_\lambda \to \mathrm{Gr}_d(E)]$ of Y_λ in $\mathrm{Gr}_d(E)$ is thus defined as the pushforward of the fundamental class of Y_λ in $\Omega^*(\mathrm{Fl}_r(U))$ along π, *i.e.*

$$[Y_\lambda \to \mathrm{Gr}_d(E)] := \pi_*[Y_\lambda \to \mathrm{Fl}_r(U)].$$

Remark 5.2. It is well-known that Y_λ is smooth and birational to X_λ along π. Since X_λ has at worst rational singularities it follows that the specialisation of the class $[Y_\lambda \to \mathrm{Gr}_d(E)]$ to $CK^*(\mathrm{Gr}_d(E))$ coincides with the fundamental class $[X_\lambda]_{CK}$ of X_λ (cf. [14]).

In order to explicitly compute $[Y_\lambda \to \mathrm{Fl}_r(U)]$, we will need the following standard result.

Lemma 5.3 ([4, Lemma 6.6.7],[16, Example 14.1.1]). *Let V be a vector bundle of rank n over X and s a section of V. Let Z be the zero scheme of s. If X is Cohen-Macaulay and the codimension of Z in X is n, then s is regular and*

$$c_n(V) = [Z \to X] \in \Omega^n(X).$$

We are finally in the position to express Y_λ as a product of Chern classes in $\Omega^*(\mathrm{Fl}_r(U))$.

Proposition 5.4. *In $\Omega^*(\mathrm{Fl}_r(U))$, we have*

$$[Y_\lambda \to \mathrm{Fl}_r(U)] = \prod_{j=1}^{r} c_{\lambda_j+d-j}\Big((D_j/D_{j-1})^\vee \otimes F_{\lambda_j+d-j}\Big). \tag{5.2}$$

Proof. For $0 \leqslant j \leqslant r$, we will denote by Y_j the subvariety of $\mathrm{Fl}_r(U)$ consisting of the points $x \in \mathrm{Fl}_r(U)$ such that

$$(D_i)_x \subseteq (F^{\lambda_i+d-i})_x \quad \text{for } i = 1, \ldots, j.$$

In other words, we only consider the conditions arising from the first j parts of the partition. Furthermore, we denote by $\iota_j : Y_j \hookrightarrow Y_{j-1}$ the obvious inclusions. We will prove by induction that

$$[Y_l \to \mathrm{Fl}_r(U)] = \prod_{j=1}^{l} c_{\lambda_j+d-j}\Big((D_j/D_{j-1})^\vee \otimes F_{\lambda_j+d-j}\Big).$$

Clearly the statement is trivially satisfied for $l=0$, since $Y_0 = \mathrm{Fl}_r(U)$.

Let us now consider the inductive step. Over Y_{l-1} we have a bundle map $D_l/D_{l-1} \to E/U$ and it is easy to see that Y_l is by definition the locus where it has rank 0. Since Y_l is smooth of codimension $\lambda_l + d - l$ in Y_{l-1}, it follows from Lemma 5.3 that the fundamental class of Y_l in $\Omega^*(Y_{l-1})$ is given by

$$[Y_l \overset{\iota_l}{\to} Y_{l-1}] = \iota_{l*}(1) = (\iota_1 \cdots \iota_{l-1})^*\Big(c_{\lambda_l+d-l}\big((D_l/D_{l-1})^\vee \otimes E/F^{\lambda_l+d-l}\big)\Big), \tag{5.3}$$

where the pull-back comes from the fact that we need to restrict all bundles to Y_{l-1}. It now suffices to make use of (5.3), of the projection formula and of the inductive hypothesis to complete the proof.

$$\begin{aligned}
[Y_l \to \mathrm{Fl}_r] &= (\iota_{1*} \circ \cdots \circ \iota_{l*})(1) = (\iota_1 \cdots \iota_{l-1})_*\Big(\iota_{l*}(1)\Big) \\
&= (\iota_1 \cdots \iota_{l-1})_*\Big((\iota_1 \cdots \iota_{l-1})^*\Big(c_{\lambda_l+d-l}\big((D_l/D_{l-1})^\vee \otimes F_{\lambda_l+d-l}\big)\Big)\Big) \\
&= c_{\lambda_l+d-l}\big((D_l/D_{l-1})^\vee \otimes F_{\lambda_l+d-l}\big) \cdot (\iota_1 \cdots \iota_{l-1})_*(1) \\
&= c_{\lambda_l+d-l}\big((D_l/D_{l-1})^\vee \otimes F_{\lambda_l+d-l}\big) \cdot [Y_{l-1} \to \mathrm{Fl}_r] \\
&= c_{\lambda_l+d-l}\big((D_l/D_{l-1})^\vee \otimes F_{\lambda_l+d-l}\big) \cdot \prod_{j=1}^{l-1} c_{\lambda_j+d-j}\Big((D_j/D_{j-1})^\vee \otimes F_{\lambda_j+d-j}\Big). \quad \square
\end{aligned}$$

We now want to compute the class $[Y_\lambda \to \mathrm{Gr}_d(E)]$ by pushing forward the product of Chern classes (5.2) through the tower of projective bundles (5.1). We first need some algebraic preparations, following [14]. Set $R = \Omega^*(\mathrm{Gr}_d(E))$, viewed as a graded algebra over $\mathbb{L}$. Let $t_1, \ldots, t_r$ be indeterminates of degree 1. We use the multi-index notation $t^{\mathbf{s}} := t_1^{s_1} \cdots t_r^{s_r}$ for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$. A formal Laurent series $f(t_1, \ldots, t_r) = \sum_{\mathbf{s} \in \mathbb{Z}^r} a_{\mathbf{s}} t^{\mathbf{s}}$ is *homogeneous of degree* $m \in \mathbb{Z}$ if $a_{\mathbf{s}}$ is zero unless $a_{\mathbf{s}} \in R_{m-|\mathbf{s}|}$ with $|\mathbf{s}| = \sum_{i=1}^r s_i$. Set $\operatorname{supp} f = \{\mathbf{s} \in \mathbb{Z}^r \mid a_{\mathbf{s}} \neq 0\}$. For each $m \in \mathbb{Z}$, define $\mathcal{L}_m^R$ to be the space of all formal Laurent series of homogeneous degree m such that there exists $\mathsf{n} \in \mathbb{Z}^r$ such that $\mathsf{n} + \operatorname{supp} f$ is contained in the cone in $\mathbb{Z}^r$ defined by $s_1 \geqslant 0,\ s_1 + s_2 \geqslant 0,\ \cdots,\ s_1 + \cdots + s_r \geqslant 0$. Then $\mathcal{L}^R := \bigoplus_{m \in \mathbb{Z}} \mathcal{L}_m^R$ is a graded ring over R with the obvious product. For each $i = 1, \ldots, r$, let $\mathcal{L}^{R,i}$ be the R-subring of $\mathcal{L}^R$ consisting of series that do not contain any negative powers of $t_1, \ldots, t_{i-1}$. In particular, $\mathcal{L}^{R,1} = \mathcal{L}^R$. A series $f(t_1, \ldots, t_r)$ is a *power series* if it doesn't contain any negative powers of $t_1, \ldots, t_r$. Let $R[[t_1, \ldots, t_r]]_m$ denote the set of all power series in $t_1, \ldots, t_r$ of degree $m \in \mathbb{Z}$. We set $R[[t_1, \ldots, t_r]]_{\mathrm{gr}} := \bigoplus_{m \in \mathbb{Z}} R[[t_1, \ldots, t_r]]_m$.

Before we proceed with the main definition and the actual computation, it is worth spending a few words about the method developed by Kazarian in [15]. The fundamental insight behind his approach is that, in the special case of the classes appearing in (5.2), the iterated pushforwards we need to perform can be reduced to formal manipulations of Laurent series. In fact, in view of Theorem 4.9, at every stage the only essential piece of information is given by the exponent of the first Chern class of the tautological bundle D_j/D_{j-1}, which gets then replaced through the push-forward by the appropriate relative Segre class (Lemma 5.6). In other words, the whole problem is turned into the identification of the Laurent series providing the correct answer.

Definition 5.5. For each $j = 1, \ldots, r$, define a graded $R[[t_1, \ldots, t_{j-1}]]_{\mathrm{gr}}$-module homomorphism

$$\phi_j : \mathcal{L}^{R,j} \to \Omega^*\big(\mathbb{P}(U/D_{j-2})\big)$$

by setting

$$\phi_j(t_1^{s_1} \cdots t_r^{s_d}) = \tau_1^{s_1} \cdots \tau_{j-1}^{s_{j-1}} \mathscr{S}_{s_j}(j) \cdots \mathscr{S}_{s_r}(r)$$

where $\mathscr{S}_m(i) := \mathscr{S}_m((U - F_{\lambda_i - i + d})^\vee)$ for $m \in \mathbb{Z}$ and $i = 1, \ldots, r$. It is known that $\Omega^*(\mathbb{P}(U/D_{j-2}))$ is bounded above, *i.e.* $\Omega^m(\mathbb{P}(U/D_{j-2})) = 0$ for all $m > \dim \mathbb{P}(U/D_{j-2})$. Therefore $\mathscr{S}_m(i)$ is zero for all sufficiently large m. This ensures that the above map is well-defined.

We have the following pushforward formula for each stage of the tower in terms of ϕ_j.

Lemma 5.6. *Let $\alpha_j := c_{\lambda_j+d-j}\big((D_j/D_{j-1})^\vee \otimes F_{\lambda_j+d-j}\big)$ for $j = 1, \ldots, r$. For each non-negative integer s, we have*

$$\pi_{j*}\big(\tau_j^s \alpha_j\big) = \phi_j\left(t_j^{\lambda_j+s} \prod_{i=1}^{j-1} (1 - t_i/t_j) P(t_j, t_i)\right).$$

Proof. By applying Theorem 4.9 to $\pi_j : \mathbb{P}(U/D_{j-1}) \to \mathbb{P}(U/D_{j-2})$ the left hand side can be evaluated as

$$\pi_{j*}(\tau_j^s \alpha_j) = \mathscr{S}_{\lambda_j+s}\big((U/D_{j-1} - F_{\lambda_j+d-j})^\vee\big) = \mathscr{S}_{\lambda_j+s}\big((U - F_{\lambda_j+d-j})^\vee - D_{j-1}^\vee\big).$$

From the definition of the relative Segre class (4.2), we obtain

$$\pi_{j*}\big(\tau_j^s \alpha_j\big) = \sum_{p=0}^{j-1} \sum_{q=0}^{\infty} (-1)^p c_p(D_{j-1}^\vee) w_{-q}(D_{j-1}^\vee) \mathscr{S}_{\lambda_j+s-p+q}(j).$$

Thus by using ϕ_j, we have

$$\begin{aligned} \pi_{j*}\big(\tau_j^s \alpha_j\big) &= \phi_j\left(\sum_{p=0}^{j-1} \sum_{q=0}^{\infty} (-1)^p e_p(t_1, \ldots, t_{j-1}) w_{-q}(t_1, \ldots, t_{j-1}) t_j^{\lambda_j+s-p+q}\right) \\ &= \phi_j\left(t_j^{\lambda_j+s}\left(\sum_{p=0}^{j-1} (-1)^p e_p(t_1, \ldots, t_{j-1}) t_j^{-p}\right)\left(\sum_{q=0}^{\infty} w_{-q}(t_1, \ldots, t_{j-1}) t_j^q\right)\right). \end{aligned}$$

The claim follows from the definitions of w_{-q} and of the elementary symmetric polynomials e_p in terms of their generating functions. ∎

Now we obtain our main application. Set

$$\Delta_{\mathbf{m}}\Big(\mathscr{S}(1), \ldots, \mathscr{S}(r)\Big) := \det\Big(\mathscr{S}_{m_i+j-i}(i)\Big)_{1 \leqslant i,j \leqslant r}$$

for each $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $a_{\mathbf{s}} \in \mathbb{L}$ be the coefficients of the power series

$$\prod_{1 \leqslant i < j \leqslant r} P(t_j, t_i) = \sum_{\mathbf{s}=(s_1,\ldots,s_r) \in \mathbb{N}^r} a_{\mathbf{s}} \cdot t_1^{s_1} \cdots t_r^{s_r}$$

viewed as an element of $\mathcal{L}^{\mathbb{L}}$.

Theorem 5.7. *For a partition $\lambda \in \mathcal{P}_d(n)$ of length r, the class $[Y_\lambda \to \mathrm{Gr}_d(E)]$ is given by*

$$[Y_\lambda \to \mathrm{Gr}_d(E)] = \sum_{\mathbf{s}=(s_1,\ldots,s_r) \in \mathbb{N}^r} a_{\mathbf{s}} \Delta_{\lambda+\mathbf{s}}\Big(\mathscr{S}(1), \ldots, \mathscr{S}(r)\Big).$$

Proof. By Definition 5.1 and Proposition 5.4, we have

$$[Y_\lambda \to \mathrm{Gr}_d(E)] = \pi_{1*} \circ \cdots \circ \pi_{r*} \left(\prod_{i=1}^{r} \alpha_i \right)$$

and a repeated application of Lemma 5.6 (*cf.* [14, Section 4.4]) yields

$$[Y_\lambda \to \mathrm{Gr}_d(E)] = \phi_1 \left(t_1^{\lambda_1} \cdots t_r^{\lambda_r} \prod_{1\leqslant i<j\leqslant r} (1 - t_i/t_j) \prod_{1\leqslant i<j\leqslant r} P(t_j, t_i) \right).$$

Since ϕ_1 is linear, we have

$$[Y_\lambda \to \mathrm{Gr}_d(E)] = \sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{N}^r} a_{\mathbf{s}} \phi_1 \left(t_1^{\lambda_1+s_1} \cdots t_r^{\lambda_r+s_r} \prod_{1\leqslant i<j\leqslant r} (1 - t_i/t_j) \right).$$

Vandermode's determinant formula allows us to write

$$t_1^{\lambda_1+s_1} \cdots t_r^{\lambda_r+s_r} \prod_{1\leqslant i<j\leqslant r} (1 - t_i/t_j) = \det \left(t_i^{\lambda_i+s_i+j-i} \right),$$

thus by applying ϕ_1 we obtain

$$[Y_\lambda \to \mathrm{Gr}_d(E)] = \sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{N}^r} a_{\mathbf{s}} \det \left(\mathscr{S}_{\lambda_i+s_i+j-i}(i) \right)_{1\leqslant i,j\leqslant r}.$$

This completes the proof. □

In connective K-theory $[Y_\lambda \to \mathrm{Gr}_d(E)]_{CK}$ coincides with the fundamental class of the degeneracy locus X_λ and thus Theorem 5.7 implies the following determinantal formula describing $[X_\lambda]_{CK}$, which is different from the one obtained in [14].

Corollary 5.8. *For a partition $\lambda \in \mathcal{P}_d(n)$, we have*

$$[X_\lambda]_{CK} = \det \left(\sum_{s\geqslant 0} \binom{i-r}{s} (-\beta)^s \mathscr{S}_{\lambda_i+j-i+s}(i) \right)_{1\leqslant i,j\leqslant r}.$$

Proof. For connective K-theory one has $P_{CK}(x,y) = \frac{1}{1-\beta y}$, so in this case the formula follows from the identity

$$t_1^{\lambda_1} \cdots t_r^{\lambda_r} \prod_{1\leqslant i<j\leqslant r} (1 - t_i/t_j) \prod_{1\leqslant i<j\leqslant r} P(t_j, t_i) = \det \left(\left(\frac{1}{1-\beta t_i} \right)^{r-i} t_i^{\lambda_i+j-i} \right)_{1\leqslant i,j\leqslant r}.$$ □

Remark 5.9. It is easy to see that when one sets $\beta = 0$ the previous equality recovers (1.2), the classical Damon–Kempf–Laksov formula for the Chow ring expressed in terms of Segre classes.

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